Ordered sets of Baire class 1 functions

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General question

Pointwise ordering

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Definition. Suppose that $(P, <_P)$ and $(Q, <_Q)$ are posets. We say that P *is embeddable into* Q, in symbols $(P, <_P) \hookrightarrow (Q, <_Q)$ if there exists a map $\Phi : P \to Q$ such that for every $p, q \in P$ if $p <_P q$ then $\Phi(p) <_Q \Phi(q)$.

Known results: Continuous case

Proposition. (Folklore) For a linearly ordered set $(\mathbb{L}, <_{\mathbb{L}})$

$$(\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow (C(X, \mathbb{R}), <) \text{ iff } (\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow ([0,1], <).$$

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The map $f \mapsto \text{subgraph}(f) = \{(x, y) : y \leq f(x)\}$ is an embedding $(C(X, \mathbb{R}), <) \hookrightarrow (\mathbf{\Pi}^0_1(X \times \mathbb{R}), \subset)$.

Known results: Continuous case

Enough:

$$(\mathbf{\Pi}^0_1(X \times \mathbb{R}), \subset) \hookrightarrow ([0,1], <).$$

Let $\{U_n : n \in \omega\}$ be a basis of $X \times \mathbb{R}$.

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Map $F \in \mathbf{\Pi}^0_1(X \times \mathbb{R})$ to $\sum_{U_n \cap F \neq \emptyset} 3^{-n-1}$.

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□

Observe that we did not use the continuity, just that the sets subgraph(f) are closed.

Definition. A function f is called *upper semicontinuous (USC)* if subgraph(f) is closed.

Known results: Higher Baire classes

Borel functions

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Baire class 2 functions

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Known results: Baire class 1

Kuratowski's theorem

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Theorem. (Komjáth, 1990) Consistently no: If $(\mathbb{S}, <)$ is a Suslin line, then $(\mathbb{S}, <) \not\hookrightarrow (\mathcal{B}_1(X), <)$.

Known results: Baire class 1

A non-characterisation result

Theorem. (Elekes, Steprāns, 2006) There exists a linear ordering $(\mathbb{L}, <_{\mathbb{L}})$ such that neither ω_1 nor ω_1^* is embeddable into $\mathbb{L}$, but $(\mathbb{L}, <_{\mathbb{L}}) \not\hookrightarrow (\mathcal{B}_1(X), <)$.

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The positive direction

Theorem. (Elekes, Steprāns, 2006) (MA) If $|\mathbb{L}| < \mathfrak{c}$ and neither ω_1 nor ω_1^* is embeddable into $(\mathbb{L}, <_{\mathbb{L}})$ then $(\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow (\mathcal{B}_1(X), <)$.

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Main Theorem. (Elekes, V.) There exists a universal linear ordering embeddable into the poset of Baire class 1 functions, i. e., a linearly ordered set $(U, <_U)$ such that for every linearly ordered set $(\mathbb{L}, <_{\mathbb{L}})$ we have

$$(\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow (\mathcal{B}_1(X), <) \text{ iff } (\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow (U, <_U).$$

The universal ordering: $([0,1]^{<\omega_1}_{\searrow}, <_{altlex})$

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Let $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0,1]^{<\omega_1}_{\searrow}$ be distinct and let δ be minimal such that $x_\delta \neq x'_\delta$. We say that $\bar{x} <_{altlex} \bar{x}' \iff$

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Main Theorem. (Elekes, V.)

$$(\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow (\mathcal{B}_1(X), <) \text{ iff } (\mathbb{L}, <_{\mathbb{L}}) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow}, <_{altlex}).$$

In fact,

$$(\mathcal{B}_1(X), <) \rightleftarrows ([0,1]^{<\omega_1}_{\searrow}, <_{altlex}).$$

About the proof

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Theorem. (Hausdorff, Kuratowski) For every $A \in \mathbf{\Delta}^0_2$ there exists a strictly decreasing transfinite sequence of closed sets $(F_\beta)_{\beta \leq \xi}$ for some $\xi < \omega_1$ such that

$$A = \bigcup_{\gamma < \xi, \gamma \text{ is even}} (F_\gamma \setminus F_{\gamma+1}).$$

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$([0,1]^{<\omega_1}_{\searrow}, <_{altlex}) \hookrightarrow (\mathcal{B}_1(X), <)$

X, X' are σ-compact then $(\mathcal{B}_1(X), <) \leftrightarrows (\mathcal{B}_1(X'), <)$.

Enough: $([0,1]^{<\omega_1}_{\searrow}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(\mathcal{K}([0,1]^2)), \subset)$.

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- Komjáth: a forcing-free proof of the non-embeddability of Suslin lines.

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- Lexicographical countable products of embeddable linearly ordered sets are also embeddable.
- Completions of a embeddable linearly ordered sets are not necessarily embeddable.

Open problems

Question. What can we say about linear orderings embeddable into the poset of Baire class α functions if $\alpha \geq 2$ in terms of universal orderings? What if we consider the poset $(\mathbf{\Sigma}^0_\alpha(X), \subset)$ for some $\alpha \geq 2$?

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Question. Does there exist a universal linearly ordered set if X is only separable metrisable?

Thank you for your attention!